Impact of Cognitive Activities Involved in Teaching Differential Equations in Secondary School

Yassine Naamaoui [1], Mohamed Chergui [2], Bouazza El Wahbi [1]

[1] *Faculty of Sciences, LAGA-Lab ,University Ibn Tofail, Kenitra, Morocco*
[2] *Regional Centre for Education and Training Professions, LaREAMI Lab, Kenitra, Morocco*

***Abstract* – This article investigates the impact of cognitive activities involved in teaching differential equations in secondary school. An analysis of textbooks is undertaken to identify the cognitive activities required from the students in learning and investing of differential equations. We noted the lack of support for the cognitive activities of changing frames, conversions of registers of semiotic representations and modeling. A test administered to a group of students at terminal science was carried out to diagnose their acquisitions on differential equations. Results showed various types of difficulties which were observed in the resolution of certain equations from the field of physics.**

***Keywords* – teaching differential equations, interplays between different frames, registers of semiotic representations, modeling.**

1. Introduction

Differential equations are a fairly important notion in mathematics that has given its contribution in the development of several theories.

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Corresponding author: Mohamed Chergui,
Regional Centre for Education and Training Professions, LaREAMI Lab, Kenitra, Morocco.
Email: chergui_m@yahoo.fr

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In certain writings on the history of mathematics, it is to them that we owe the emergence of modern analysis [1]. Culturally, it is a concept well recognized by its contribution in the mathematical representation of various phenomena giving the possibility of describing them, analyzing them and conjecturing their mode of evolution.

Given this importance, the teaching of differential equations is integrated into several secondary school curricula. However, their transposition in different teaching cycles has revealed difficulties in teaching practices and in the processes of conceptualization by learners. This is reported in several research works which have focused more particularly on the domination of the algebraic approach and the place of modeling in the teaching of differential equations. About these, one can consult for example the references [2], [3] and [4].

It should also be noted that the interdisciplinary nature of this notion places it in the didactic continuity [5] between mathematics and other disciplines. As a result, the cognitive issues underlying the teaching of differential equations have to be appropriately identified at all levels.

In this dynamic of reflection on these issues, we are interested in this research of the impact of the cognitive activities involved in the teaching of differential equations in secondary school on the learning and the investment of this notion. Thus, we first identified the cognitive activities evoked in textbook situations involving differential equations. To determine the impact of the choices made, we examined the productions of a group of students on a test relating to this concept.

Our article is divided into four sections. The first focuses on a review of previous works, the second explains the problem studied and the methodological framework. Our results are the subject of the third section, followed by discussion. We end with a summary of the results of this work.

2. Literature Review

In many countries, several researches on teaching and learning differential equations has focused on the negative effects of neglecting numerical and graphical methods in solving differential equations. This observation of the dominance of the algebraic approach has been reported by Artigue [6] in France, by Rasmussen [7] in the United States, for Mexico by Ramirez [8] and for Indonesia in [9]. Blanchard et al. in the preface to their book on differential equations [10] called for a change of the classical teaching of differential equations.

By asking the question of the choices for teaching of differential equations in the mathematical institution and their effects on the construction of student knowledge, Saglam-Arslan [11] observed, with a group of French students, that differential equations were only used to overcome the questions encountered through an algorithmic approach without taking into account their major role in the process of modeling the phenomena studied. This was also deduced by Khotimah and Masduki [9] in a study on the process of learning differential equations at university. They noticed that most students were not able to combine the concepts learned in differential and integral calculus to solve these problems. Gordillo [4] noted difficulties in the articulation of graphic and symbolic registers in students who were preparing for the Certificate of Aptitude for Teaching in Secondary Education.

Faced with these cognitive dysfunctions and skills deficits, alternatives in the teaching of differential equations started to emerge. In this context, Arslan [12], by carrying out didactic engineering, was able to deduce that it is possible to establish a qualitative approach to differential equations for the final classes. In higher education, the same author in collaboration with C. Laborde sought to invest the computer tool to promote the interaction between algebraic and graphical frameworks to overcome the difficulties of learning the qualitative approach [13].

In parallel with these didactic efforts, work with a cognitive scope emerged. Leão et al. [14] talk about a new pedagogical paradigm that emphasizes the role of learning in modeling, analyzing the differential equation, understanding the qualitative behavior of the solution and finally being able to communicate mathematically instead of performing tasks or methods. In this vision, the authors in [15] were able to conclude that the implementation of contextual learning in the course of differential equations allows the improvement of reasoning skills in students, manifested by a growth in terms of analyzing the problem, organizing the solution, interpreting the results and explaining the problem at hand.

This review shows that for an efficiency of knowledge in differential equations, the cognitive functions engaged in their teaching have to be well specified.

3. Theoretical Framework

In this section, and in the light of the previous study, we will highlight the cognitive and didactic place of modeling, interplays between different frames and conversion of registers of semiotic representations.

According to Chevallard [16], modeling is a process consisting of the following three steps:

- We define the system that we intend to study, by specifying the relevant aspects in relation to the study that we want to make of this system, i.e. the set of variables by which it is divided into the domain of reality where it appears to us.
- The model is built by establishing a certain number of relations, R, R', R'', etc., between the variables taken into account in the first stage.
- We work on the model thus obtained, with the aim of producing knowledge relating to the system studied, knowledge which takes the form of new relations between the variables of the system.

For Thorn [17], the integration of applications and modeling in the curricula helps to give more meaning to the learning and teaching of mathematics. According to S. P. Carreira [18] it is the concept of meaning that gives another perspective on the role of mathematical modeling and applications: students become able to understand mathematical concepts insofar as they will give meaning to mathematics.

It turns out that the conceptualization and access to the meaning of concepts go through an interaction between the fields of mathematics and those of extra-mathematics via applications and modeling.

Cognitively, this interaction manifests itself mainly through a set of frames and through a diversified semiotic practice. Let us take a closer look at these two concepts.

For R. Douady [19] a frame has a broad meaning, it can designate a mathematical domain or a field of knowledge that does not belong to mathematics, such as the frames of physics, chemistry and etc. Interplay between different frames is changes of frames intentionally provoked by the teacher, on the occasion of suitably chosen problems, to advance the phases of research and to evolve the conceptions of the pupils. The interplay between different frames [19] makes it possible to obtain different formulations of a problem and promotes new access to the difficulties encountered and the implementation of tools and techniques that were not essential in the initial formulation. It is therefore a very important cognitive activity since these changes of frames lead the learner to mobilize his prerequisites in situations qualified by R. Douady as dialectic such as object/tool or old/new.

On the other hand, the externalization of learning or the communication of productions carried out in writing or orally is an essential task in the learning process and to carry it out, representations of the objects in play are necessary. In this respect, Duval [20] introduces the notion of register of semiotic representation as being "… productions constituted by the use of signs belonging to a system of representations which has its own constraints of significance and functioning, the set of these signs is called the register of semiotic representation". He emphasizes the importance of semiotic representations in the manipulation of abstract mathematical objects. In general, a mathematical object uses many registers of semiotic representations and each of them provides partial access to the object represented and allows certain operations to be performed on this object. Duval considers that the coordination of several registers of the same notion is fundamental for its conceptualization. This can be done through the implementation of three fundamental cognitive activities, namely the formation of a representation identifiable as a representation of a given object, the treatment of a representation in the same register in which it was formed and the conversion of a representation into another of another register.

According to Duval, a lack of mastery or relevance in the registers or in the formation, processing and conversion operations during teaching can have a negative impact on the formation of mental representations in students.

It is important to signal that the change of frame is distinct from the change of register called "representation conversion" by Duval [20]. The change of frame consists of "translating a problem into a field of work other than the one that the first presentation of the problem makes it possible to identify",while the change of register consists in the passage for example from a figure to a statement, or from a statement in English to an algebraic formula. There can therefore have register changes without there being a frame change mathematical.

The interest of changing frames and registers is also present in the theory of conceptual fields elaborated by Vergnaud [21], which considers that any concept is characterized by the set of situations which give it meaning, operational invariants of the concept and a signifier which is the set of linguistic forms or not, necessary to represent it symbolically and makes its processing feasible.

To end the current section, it is valuable to note that the concept of differential equations works in several frames: algebraic, numerical and geometric. It also admits several registers of representation: natural language, algebraic expressions of equations and solutions, solution curves, tangent fields, numerical tables, etc. Working in the same frame can call upon several registers of representation and the change of frames necessarily implies passages between registers.

4. Problem Statement and Methodology

In this section, we present the problem studied, expose the research questions and we explain the methodological choices.

4.1. Problem Statement

The previous study reveals that the management of changes in frames and registers of semiotic representations and implementing them in the process of teaching and learning differential equations is essential to access the meaning of this notion and to acquire the skills necessary for modeling by this notion. As a result, this study investigates the impact of the cognitive activities involved in the teaching of differential equations in secondary school on the learning and investment of this notion.

Our objective is to know if the cognitive functions proposed in the teaching of differential equations contribute favorably to a good apprehension of this notion and to an ease of its investment. In connection with this problem, we formulate the following questions:

- What cognitive functions are involved in the activities of mathematics textbooks?
- Does the teaching provided for differential equations in secondary school promote their investment in situations that are not related to mathematics?

Before explaining the methodological choices that allow us to answer the questions posed, let us first define the institutional framework of the notion studied.

This exploratory study will focus on the case of the teaching of differential equations in secondary school in Morocco where the differential equations are presented for the science terminal classes (17-18 years old) after the teaching of primitive functions and the exponential function. The targeted capacities are the resolution of the differential equations of the two types $y' = ay + b$ and $y" + ay' + by = 0$ and of those which are reduced to these two types.

According to the official instructions, provided at the beginning of both textbooks presented in Table 1, it is recommended to teachers that the differential equations have to be invested in situations relating to the field of physics or other without this ability being the subject of an evaluation.

In terminal classes, the students' relationship to differential equations is also established in physics. In fact, official guidelines recommend the need for learners to acquire the ability to model the phenomena studied. Among the rather invested mathematical models, the notion of differential equations is used to model the behavior of a capacitor and a coil and to analyze and conjecture the evolution of a mechanical system. Let us note here that differential equations that are not necessarily linear and graphical and numerical methods are also used.

4.2. Methodology

To answer the questions asked, we adopt a qualitative approach based on the conclusions of the literature review conducted. For the first question, we analyzed all the activities proposed on differential equations for the final classes in the two mathematics textbooks accredited by the competent services of the Ministry of National Education and intended for experimental sciences and technical sections. The data of these two textbooks are presented in the Table 1.

Table 1. Identification of Texbooks analyzed

Texbooks	**Ministerial accreditation number**
Texbook 1	09CB21207
Texbook 2	09CB21307

In each textbook, the chapter on differential equations is composed of three parts. The first is devoted to preparatory activities for the introduction of new knowledge, the second is dedicated to the institutionalization of knowledge and the last part is reserved to activities investing new learning.

Our analysis focuses on preparatory and investment activities, whose respective numbers are 8 and 39.

Based on the bibliographic study carried out, our analysis aims to identify the cognitive activities involved in terms of interplay of frames, conversion of registers of semiotic representations and modeling. Therefore, preparatory and investment activities will be analyzed according to the following two criteria:

- The frames used for formulating the activity and those required to produce the responses expected from the instructions;
- The activity formulation registers and those necessary to produce the required responses.

The analysis will be carried out on the basis of the grid, in Table 2, which gives indicators to identify in the official texts and in each activity of the textbooks the frames and registers involved. The development of this grid referred to the theoretical framework developed in this paper.

Table 2. Analysis grid of activities according to frames and registers

Types of frames and registers	Indicators to identify the frame or the register involved
Algebraic	The use of algebraic methods
Geometric	The use of geometric concepts
Graphic	The use of curves and related methods
Frames from another discipline	Physics, biology, etc.
Graphic register	The use of graphical terminology
Algebraic register	The use of function expressions, formulas, etc.
Symbolic register	The use of symbols and in particular of ostensives.

For the second question, a test aimed at solving differential equations is administered during the 2020-2021 school year, to a group of students in the final year of experimental sciences. The test (Table 3) is composed of two situations from the field of physics. In each situation, the student is asked to solve a differential equation.

Table 3. Test administred

Situation 1 : The speed v, as a function of time t, of an object in free fall in the atmosphere is given by the equation: $(\mathbf{E_1})\quad \mathbf{m\frac{dv}{dt} = mg - \gamma v}$; $\mathbf{m}, \mathbf{g}$ and $\boldsymbol{\gamma}$ are constants. Solve the equation $(\mathbf{E_1})$.
Situation 2 : The charge Q(t) applied to the capacitor of a circuit equipped with a capacitance C, a resistance R and an inductance L is given by the equation, $\mathbf{L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)}$. Solve the following equation $(\mathbf{E_2})$: $\mathbf{L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = 0}$.

The a priori analysis of this test is presented in Table 4 below:

Table 4. A priori analysis of the test

	Situation 1	Situation 2
Type of the equation proposed	$y' = ay + b$	$y'' + ay + by = 0$
Identification of the desired solution	The speed v as a function of time	The electric charge Q as a function of time
Frames used in the situation	Mechanics and Algebraic	Electricity and algebraic
Semiotic registers	Algebraic and symbolic	
Possible method for resolution	M_1: Restitution the expression of the solution provided in mathematics class. M_2: Finding the solution using primitive functions.	Restitution of the expression of the solution provided in mathematics class.

Before the administration and for validation purposes, the test was presented for 3 teachers of mathematics and 3 others of physics who had already taught final classes. The test is distributed during the first two weeks of May 2021 in high schools belonging to the Regional Academy of Education and Training of Rabat-Sale-Kenitra for students who have studied the courses on the knowledge involved in the two test situations.

Given the constraints imposed by the COVID- 19 pandemic on schooling modes, manifested by the reduction in the number of pupils in face-to-face education and by the delay in the progress of program implementation, the number of students who participated in the test could only reach 64 spread over 6 secondary schools.

The analysis of the productions of the pupils tested concerns the following aspects which are related to the ability to invest learning in differential equations:

- Recognition of the type of differential equation put into situation;
- Identification of the nature of the solution of the differential equation put into situation (the speed v for the first equation and the electric charge Q for the second according to time t);
- Steps taken to determine the solution;
- Veracity of the solution obtained;
- Semiotic aspects.

5. Results

5.1. Analysis of Textbook Activities

The analysis of the eight activities in the chapter on differential equations revealed that seven are formulated in the algebraic frame and only one is formulated in the physics frame, but all of them just require the algebraic frame to produce the answers to the given instructions. For the registers of formulation of the statements of the 8 activities and those required to formulate the answers, there are algebraic and symbolic registers.

The results of the comprehensive examination of investment activities according to the frames and registers that are included in the statements or necessary to formulate the solutions are presented in Table 5.

Table 5. Classification of investment activities according to frames and registers

Frame or register involved	In formulation of the statement	Necessary for the production of answers
Algebraic frame	47	59
Graphic frame	8	0
Physics	8	0
Biology	4	0
Algebraic register	56	59
Symbolic register	59	59

5.2. Test Results

The analysis of the productions of the 64 pupils who took part in the test is recorded in Table 6 below.

Table 5. Results of the test

Recognition of the type of differential equation		Identification of the nature of the solution		Approach carried out in the determination of the solution of the equation (E_1)	
Yes	No	Yes	No	M_1	M_2
56	8	46	18	12	52

Concerning the veracity of the productions made by the pupils tested the quantitative results are summarized in Table 7.

Table 6 . Classification of answers according to veracity

Correct answers		False answers	
(E_1)	(E_2)	(E_1)	(E_2)
20	8	44	56

Regarding these results, let us note the following:

- 18 pupils, who did not identify the nature of the solution, have provided *y* in terms of *x*.
- One of the main reasons behind the inability to produce the right solution is the difficulty in using the coefficients involved in the two

differential equations. For equation (E_1), 32 pupils among those who chose method M_2 were unable to determine the primitive sought due to the confusion generated when switching from the coefficients a and b used in mathematics to those proposed in the equation.

- In the resolution of the equation (E_2), 50 pupils were unable to write the characteristic equation correctly.
- Among the pupils who succeeded in solving the equations correctly, 15 pupils used the writing $mv' = mg - \gamma v$ for (E_1) and $LQ'' + RQ' + \frac{1}{C}Q = E$ by 6 pupils for (E_2).

6. Discussion

Examination of the preparatory activities for conceptualization proposed in textbooks proves that they have the exclusive function of communicating to the student the algebraic expressions which will be referred to later in the course as differential equations and their explicit solutions. Consequently, the action of the learner is reduced to performing algebraic manipulations on these solutions and the learner differential equation relationship is only established in the context of computational activities. This is also very clear in the dominant use of the algebraic frame and register in the formulation of preparation activities.

The textbook also has the function of proposing investment situations by suggesting to the learner to use his knowledge in a varied series of situations. The examination carried out on the investment activities allows us to conclude that this function is not carried out by the two textbooks. Indeed, the exhaustive inventory of frames and registers brought into play in this type of activity confirms the dominance of the use of the algebraic frame and register for the formulation of investment situations and for the enunciation of the expected responses.

In Textbook 1, only two exercises refer to the field of physics and 4 others to biology. The use of graphical or geometric frames only takes place to present the initial data of a differential equation. Similarly, extra-mathematical frames are only mentioned for information on the origin of the problem without a task related to the modeling process being targeted.

This negligence in engaging the modeling process and in diversifying frames and registers in the development of textbooks is not adapted to the fairly diversified situations programmed in physics.

On the other hand, on the semiotic level, each parameter used in a differential equation in physics refers to a quantity. These are ostensives R, C, u, q, i, etc. which establish a clear correspondence between signifiers and signifieds. Therefore, the use of signs is placed at the service of the production of meaning. This again attests to the inconsistency between the teaching of differential equations in mathematics and their modes of investment in physics.

In terms of communication, the dominance of the algebraic treatment of differential equations weighs on the development of communication in mathematics, in particular the description and interpretation of phenomena from fields of science and formulating positions with clear and precise language. Let us note here the high number of students who could not identify the nature of the required solution, instead of finding the speed v and the electric charge Q as a function of time, 18 students provided the solutions by y as a function of x. This inability to communicate with the language specific to the domain studied confirms the negative effect of not putting the learner in a situation of practice of several registers of semiotic representations during learning. This is also confirmed by the use of several learners of common notations in mathematics for derivative functions.

On the other hand, cognitive choices in the teaching of differential equations have a negative impact on the ability of students to invest their knowledge of differential equations in situations arising from extra-mathematical fields. Indeed, according to the results of the test, 46 pupils, among the 56 who recognized the type of differential equation involved, did not succeed in writing the exact expression of the solution. In the resolution of the equation (E_1) and among those who have chosen to seek its solution by primitive functions, difficulties in writing the original equation in the form $y' = ay$ have been observed and a confusion in the identification of the constants a and b among those who have chosen to substitute directly by the general solution $y = \alpha\, e^{ax} - \frac{b}{a}$. In both cases, it is indeed a question of semiotic difficulties.

For the equation(E_2), the failure in the determination of the solution is mainly due to the erroneous writing of the characteristic equation and to the arbitrary choice of the expression of the solution without specifying its conditions of existence which must be established according to the sign of the discriminant of the characteristic equation.

7. Conclusion

This research was conducted to explore the impact of cognitive activities involved in the teaching of differential equations in high school. The analysis of the cognitive activities required in the activities of preparation for the conceptualization and investment of the differential equations proposed in the textbooks, makes it possible to identify the following:

- The algebraic framework is dominant in the different situations of textbooks. Indeed, the contents and capacities targeted are quite focused on computational activities based on restitution.
- The register of semiotic representations implemented in a majority way is the algebraic register with a neglect of symbols belonging to other disciplines.

It follows that the cognitive activities of changing frames and converting registers are not supported in the implementation of the program in textbooks.

We should also point out the state of inconsistency noted in this study between the teaching objectives of differential equations in mathematics and the skills necessary for the study of programmed phenomena in physics. This inconsistency concerns the types of equations involved, the resolution methods and their semiotic representations. This state is the automatic result of the absence of the implementation of modeling by differential equations in the mathematics program.

To determine the impact of this type of teaching, we administered a test where pupils were asked to solve two differential equations that model two physical phenomena. The analysis of the productions of the pupils who took part in the test showed that a considerable number of them could not find the solutions of the equations because of difficulties in the steps taken for some and for others given the difficulties of a semiotic nature manifested mainly by a clear confusion in the manipulation of notations.

In the end, we can conclude that the fact of focusing the teaching of differential equations on the activities of memorization and restitution with an absence of support for the activities of changing frames and conversions of registers of semiotic representations, implies dysfunctions in computational tasks, in the formation of exact mental representations on this notion in particular its place in the modeling of natural phenomena.

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